Converting ϵ-NFA to NFA

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1 Overview

We have observed nondeterminism in two slightly different forms in our discussion of finite automata:

- It is most apparent if there is a state q and an alphabet symbol σ such that several different transitions are possible in state q on input σ.
- A choice of moves can also occur as a result of ϵ-transitions, because there may be states from which the automata can make either a transition on an input symbol or one on no input.

We will see in this session that the second type of nondeterminism can be eliminated.

2 Converting an ϵ-NFA to an equivalent NFA

Theorem 1. *For every language L accepted by an ϵ-NFA, there is an equivalent NFA that also accepts L.*

Proof Idea

The idea is to introduce new transitions so that we no longer need ϵ-transitions: In every case where there is no σ-transition from p to q but the NFA can go from p to q by using one or more ϵ's as well as σ, we will introduce the σ-transition.

The resulting NFA may have even more nondeterminism of the first type than before, but it will be able to accept the same strings without using ϵ-transitions.

Proof

As we have already mentioned, we may need to add transitions in order to guarantee that the same strings will be accepted even when the ϵ-transitions are eliminated.
Before delving into details, let's recall the *extended transition function* for an ϵ-NFA.

Definition 1. *Let $M = (Q, \Sigma, \delta, q_0, F)$ be an ϵ-NFA. We define the extended transition function $\hat{\delta} : Q \times \Sigma^* \to 2^Q$ as follows:*

1. *For every* $q \in Q$, $\hat{\delta}(q,\epsilon) = \mathsf{ECLOSE}(q)$.

2. *For every* $q \in Q$, *every* $x \in \Sigma^*$ *and every* $\sigma \in \Sigma$, $\hat{\delta}(q, x\sigma)$ *is defined recursively as follows:*

   (a) *Let* $\hat{\delta}(q,x) = S = \{p_1, p_2, \ldots, p_k\}$.

   (b) *Compute* $\bigcup_{i=1}^{k} \delta(p_i, \sigma)$*; let this set be* $\{r_1, r_2, \ldots, r_m\}$.

   (c) *Then* $\hat{\delta}(q, x\sigma) = \mathsf{ECLOSE}(\{r_1, r_2, \ldots, r_m\})$.

Recall that the extended transition function for an NFA is defined similarly, by ignoring the ECLOSE.
Before sketching the proof, we state a useful lemma which its proof is left to the reader.

Lemma 1. *For every* ϵ*-NFA, every state* q *and strings* x, z,

$$\hat{\delta}(q, xz) = \bigcup \{\hat{\delta}(p, z) \ : \ p \in \hat{\delta}(q, x)\}$$

Proof. Given an ϵ-NFA

$$E = (Q, \Sigma, \delta_E, q_0, F_E),$$

we construct an NFA

$$N = (Q, \Sigma, \delta_N, q_0, F_N),$$

as follows, where δ_E and F_E are defined as follows:
For every $q \in Q$ and every $\sigma \in \Sigma$

$$\delta_N(q, \sigma) = \hat{\delta}_E(q, \sigma),$$

Finally, we define the accepting states of N to be:

$$F_N = \begin{cases} F_E \cup \{q_0\} & if\ \epsilon \in L(E) \\ F_E & if\ \epsilon \notin L(E) \end{cases}$$

The point of our definition of δ_N is that we wish $\hat{\delta}_N(q_0, x)$ to contain a final state if and only if the set $\hat{\delta}_E(q_0, x)$ contains a final state, for evey string $x \in \Sigma^*$. This may not be true for $x = \epsilon$, because $\hat{\delta}_E(q_0, \epsilon) = \mathsf{ECLOSE}(q_0)$ and $\hat{\delta}_N(q_0, \epsilon) = \{q_0\}$; this is the reason for the definition of F_N above. We first prove that for every state q and for every string $x \in \Sigma^*$ with $|x| \geq 1$, it holds that:

$$\hat{\delta}_N(q, x) = \hat{\delta}_E(q, x). \tag{1}$$

The proof is by structural induction on x.
Basis: If $x = \sigma \in \Sigma$, then by definition of δ_N,

$$\delta_N(q, \sigma) = \hat{\delta}_E(q, \sigma)$$

also, by definition of the extended transition function of an NFA, we have:

$$\hat{\delta}_N(q, \sigma) = \delta_N(q, \sigma).$$

Thus, we have proved that (1) holds for every string x of length one.
IH: Suppose $w = x\sigma$ where σ is the final symbol of w and assume that the

statement holds for x. That is $\hat{\delta}_N(q,x) = \hat{\delta}_E(q,x)$.
Let both these sets of states be $\{p_1, p_2, \dots, p_k\}$.
We have:

$$\begin{aligned}
\hat{\delta}_N(q,w) &= \bigcup_{i=1}^{k} \delta_N(p_i,\sigma) && \text{By the definition of the extended transition function for an NFA}\\
&= \bigcup_{i=1}^{k} \hat{\delta}_E(p_i,\sigma) && \text{Because of the definition of } \delta_N\\
&= \hat{\delta}_E(q,w) && \text{Because of the definition of } \hat{\delta}_E \text{ and lemma (1)}
\end{aligned}$$

Let's prove that $L(N) = L(E)$:

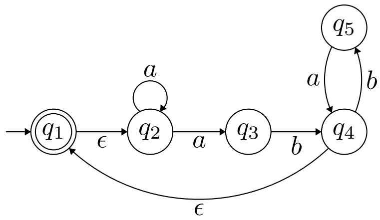
- If the string ϵ is accepted by E, then it is accepted by N, because in this case $q_0 \in F_N$ by definition. If $\epsilon \notin L(E)$, then $F_E = F_N$; therefore, $q_0 \notin F_N$, and $\epsilon \notin L(N)$.

- Suppose that $|x| \geq 1$.
  If $x \in L(E)$, then $\hat{\delta}_E(q_0,x)$ contains an element of F_E; therefore, since $\hat{\delta}_E(q_0,x) = \hat{\delta}_N(q_0,x)$ and $F_E \subseteq F_N$, $x \in L(N)$.
  Now suppose $|x| \geq 1$ and $x \in L(N)$. Then $\hat{\delta}_N(q_0,x)$ contains an element of F_N. The state q_0 is in F_N only if $\epsilon \in L(E)$; therefore, if $\hat{\delta}_N(q_0,x)$ (which is the same as $\hat{\delta}_E(q_0,x)$) contains q_0, it also contains every element of F_E in $\mathsf{ECLOSE}(q_0)$. In any case, if $x \in L(N)$, then $\hat{\delta}_N(q_0,x)$ must contain an element of F_E, which implies that $x \in L(E)$.

□

3 An Example

The following figure shows the transition diagram for an ϵ-NFA E.



We show in tabular form the values of the transition function δ_E, as well as the values $\hat{\delta}_E(q,a)$ and $\hat{\delta}_E(q,b)$ that will give us the transition function δ_N in the resulting NFA N.

The following figure shows the NFA N, whose transition function has the values in the last two columns of the above table. In this example, the initial state of E is already an accepting state, and so drawing the new transitions and eliminating the ϵ-transitions are the only steps required to obtain N.

q	$\delta_E(q,a)$	$\delta_E(q,b)$	$\delta_E(q,\epsilon)$	$\hat{\hat{\delta}}_N(q,a)$	$\hat{\hat{\delta}}_N(q,b)$
1	$\emptyset$	$\emptyset$	$\{2\}$	$\{2,3\}$	$\emptyset$
2	$\{2,3\}$	$\emptyset$	$\emptyset$	$\{2,3\}$	$\emptyset$
3	$\emptyset$	$\{4\}$	$\emptyset$	$\emptyset$	$\{1,2,4\}$
4	$\emptyset$	$\{5\}$	$\{1\}$	$\{2,3\}$	$\{5\}$
5	$\{4\}$	$\emptyset$	$\emptyset$	$\{1,2,4\}$	$\emptyset$