# Positive but not Completely Positive Maps

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# Introduction

# Quantum Mechanics in a Nutshell (1)

Quantum mechanics is a mathematical framework for describing the laws of quantum physics.
This framework is formulated using the language of dynamical systems.
To specify this dynamical system we need to specify

- its state space,
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> Keep in mind that any physical system is associated with a Hilbert space $\mathcal{H}$, over the field of complex numbers.

In our cases, you can think of these Hilbert spaces as $\mathbb{C}^d$.

# Quantum Mechanics in a Nutshell (2)

How can we formulate a composite system, consisting of two subsystems $A$ and $B$, corresponding to Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$?

The Hilbert space associated to the composite system will be

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

Remember that:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

# Quantum Mechanics in a Nutshell (3)

For a physical system associated with a Hilbert space $\mathcal{H}$, a state of the system is a matrix $\rho : \mathcal{H} \to \mathcal{H}$, such that:

1. $\text{tr}(\rho) = 1$
2. $\rho$ is positive semidefinite (PSD), denoted as $\rho \geq 0$.

These matrices are called *density matrices (operators)*.

# Quantum Mechanics in a Nutshell (4)

Dynamics of our system should be defined by superoperators $\Phi : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$.

**Definition**

A superoperator $\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is called **positive**, if for any $\rho \in \mathcal{L}(\mathcal{H}_A)$ with $\rho \geq 0$, $\Phi(\rho) \geq 0$.

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**Definition**

A superoperator $\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is called **completely positive**, if for any Hilbert space $\mathcal{H}_C$ and for all $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_C)$ with $\rho \geq 0$,

$$(\Phi \otimes \mathcal{I}_C)(\rho) \geq 0.$$

# Completely Positive Maps

Is it hard to check whether a map is completely positive?

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Is it hard to check whether a map is completely positive?

No!

Using the **Choi matrix**:

$$C_\Phi := \begin{pmatrix} \Phi(|0\rangle\langle 0|) & \Phi(|0\rangle\langle 1|) & \dots & \Phi(|0\rangle\langle d-1|) \\ \Phi(|1\rangle\langle 0|) & \Phi(|1\rangle\langle 1|) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(|d-1\rangle\langle 0|) & \dots & \dots & \Phi(|d-1\rangle\langle d-1|) \end{pmatrix}$$

**Theorem**

*$\Phi$ is completely positive iff $C_\Phi$ is PSD.*

# Quantum Entanglement (Pure States)

A simple observation:
For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, there exists a vector $v \in \mathcal{H}_1 \otimes \mathcal{H}_2$, such that

$$v \neq v_1 \otimes v_2,$$

for all $v_1 \in \mathcal{H}_1$ and $v_2 \in \mathcal{H}_2$.
For example

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \neq \begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} z \\ t \end{pmatrix}.$$

Such vectors are called **entangled**,
and anything that is not entangled is called **separable**.

# Quantum Entanglement (Density Matrices)

**Definition**

A state $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ of a composite system is called **separable** if it can be written as

$$\rho_{AB} = \sum_i p_i \sigma_i \otimes \tau_i,$$

where $\sigma_i \in \mathcal{L}(\mathcal{H}_A)$ and $\tau_i \in \mathcal{L}(\mathcal{H}_B)$ are density matrices for all $i$'s, and $p_i$'s are positive real numbers with the condition that $\sum_i p_i = 1$.
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A state that is not separable, is called **entangled**.

For example, the following state in $\mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is entangled.

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

# SEP

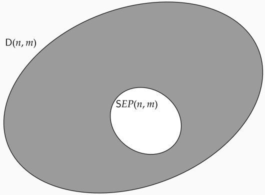
The set of all density matrices over $\mathcal{H}_A \otimes \mathcal{H}_B$ is denoted by

$$D(n, m),$$

and the set of all separable states by

$$\mathrm{SEP}(n, m),$$

where $n$ and $m$ are the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.



# Recognition of Separable States

**Theorem**

*It is* NP*-hard to determine whether an arbitrary quantum state within an inverse polynomial distance from* SEP *is entangled.*

# Optimizations over SEP

There are scenarios that we need to do an optimization over SEP.
For example, to measure the amount of entanglement of a state, we want to compute the following quantity:

$$E_R(\rho_{AB}) := \min_{\sigma_{AB} \in \mathrm{SEP}} S(\rho_{AB} \| \sigma_{AB}),$$

where $S(\rho_{AB} \| \sigma_{AB})$, which is called the *relative entropy of the states $\rho_{AB}$ and $\sigma_{AB}$* is defined as $S(\rho_{AB} \| \sigma_{AB}) := \mathrm{tr}(\rho_{AB} \log \rho_{AB}) - \mathrm{tr}(\rho_{AB} \log \sigma_{AB})$.

What can we do?

# Entanglement Detection

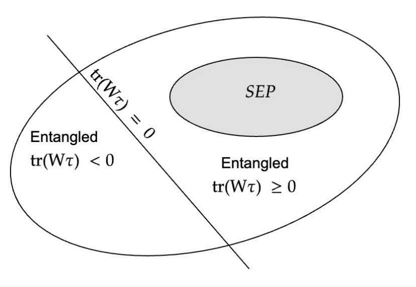
# Entanglement Criteria (1)

A simple geometric observation:

- Equip D with the inner product $\langle A, B \rangle := \mathrm{tr}(AB^\dagger)$.
- SEP is a closed convex set.

Thus, for any $\rho \notin$ SEP, we can separate $\rho$ from SEP using a hyperplane. In other words, there exists a Hermitian operator $W \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\mathrm{tr}(W\rho) < 0$ and $\mathrm{tr}(W\sigma) \geq 0$ for all separable states $\sigma$.

*W* is called an **entanglement witness**.



# Entanglement Criteria (2)

A simple algebraic observation:
Let $\Phi : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$ be a positive but not completely positive map.

- For any separable state $\rho = \sum_i p_i \sigma_i \otimes \tau_i$,

$$(\mathcal{I}_A \otimes \Phi)\rho = \sum_i p_i \sigma_i \otimes \Phi(\tau_i) \geq 0.$$

- Define the set of all states that are **p**ositive under **p**artial application of $\Phi$ by

$$\mathrm{PP}\Phi := \{\rho_{AB} \in \mathrm{D} \ : \ (\mathcal{I}_A \otimes \Phi)\rho_{AB} \geq 0\}.$$

  The above set is a proper subset of D.
- And, SEP is contained in PPΦ.

Therefore, any positive but not completely positive map provides an entanglement criterion.

# Properties of the Two Families of Criteria

# Some Important Questions

- Do we know examples of positive but not completely positive maps?
- Is it possible to describe SEP with PPΦs?
- Can we compare two criteria?
- Are PPΦs nested or incomparable?
- Are entanglement witnesses and positive but not completely positive maps related?
- Can we characterize PPΦs by entanglement witnesses?

# Examples of Positive but not Completely Positive Maps

- The most important example is the **transpose map**, whose corresponding PPΦ is known as PPT.
- The **reduction map**:

$$\Lambda(\rho) := \mathrm{tr}(\rho)\mathbb{I}_{\mathcal{H}} - \rho$$

- **Breuer-Hall maps**:

$$T_{BH}(\rho) = \mathrm{tr}(\rho)\mathbb{I}_{\mathcal{H}} - \rho - U\rho^{T}U^{\dagger},$$

  for any $U : \mathcal{H} \to \mathcal{H}$ with $U^{T} = -U$ and $U^{\dagger}U \leq \mathbb{I}$.
- The **Choi map**:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{22} \end{pmatrix}.$$

# Describing SEP with PPΦs

**Theorem**

*A state* $\rho_{AB}$ *is separable iff for all positive maps* $\Phi : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$, $\rho_{AB} \in \mathrm{PP}\Phi$.

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**Theorem**

*A state $\rho_{AB}$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable iff its partial transposition is PSD, but in higher dimensions, $(\mathcal{I} \otimes \theta)\rho_{AB} \geq 0$ is not a sufficient condition for separability.*

# Are PPΦs nested?

In high dimensions, **No!**

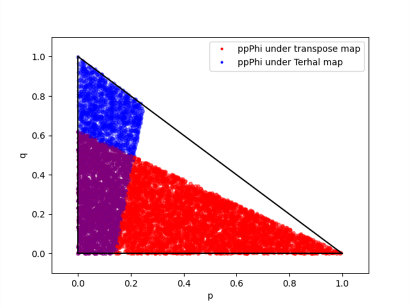


**Figure 1:** The set of states $\rho = (1 - p - q)\sigma_1 + p\sigma_2 + q\sigma_3$ and its intersection with PPΦ and PPT.

# Relationships between Witnesses and Maps

- Via **Choi–Jamiołkowski isomorphism**:

$$\mathcal{C}(\Phi) = \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes \Phi^*(|i\rangle\langle j|),$$

For a state $\rho$, if $\mathrm{tr}(\mathcal{C}(\Phi)\rho) < 0$, then, $\rho \notin \mathrm{PP}\Phi$.

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- Via **semidefinite programming**:

  Consider

$$\begin{aligned} \max_t \quad & t \\ s.t. \quad & (\mathcal{I}_A \otimes \Phi)\rho_{AB} \geq t\mathbb{I}_{AB}. \end{aligned}$$

  The dual gives us a witness:

$$\begin{aligned} \min_W \quad & (W(\mathcal{I}_A \otimes \Phi)\rho_{AB}) \\ s.t. \quad & (W) = 1, \\ & W \geq 0\,. \end{aligned}$$

$(\mathcal{I}_A \otimes \Phi^*)W^*$ is an entanglement witness.

# Characterizing PPΦ by Entanglement Witnesses

**Theorem**

*For any positive but not completely positive map* $\Phi : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$*, the family of witnesses of the form*

$$W_{|\psi\rangle} = (\mathcal{I} \otimes \Phi^*) |\psi\rangle \langle\psi| ,$$

*where* $|\psi\rangle$ *is an entangled vector in* $\mathcal{H}_A \otimes \mathcal{H}_A$*, are enough to describe* PPΦ*, meaning that* $\rho_{AB} \notin$ PPΦ *if and only if there exists a member of this family witnessing* $\rho_{AB}$*.*

**Theorem**

*Let* $W \in \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ *be a SDP witness obtained from the transpose map with an initial state* $\rho \in \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$*. Then* $W$ *is of the form*

$$W = (\mathcal{I} \otimes \theta) |\psi\rangle \langle\psi| ,$$

*where* $\theta$ *is the transpose map and* $|\psi\rangle$ *is an entangled vector in* $\mathbb{C}^2 \otimes \mathbb{C}^2$*.*

# Characterizing SDP Witnesses

**Theorem**

*Let $W$ be the SDP witness obtained from the transpose map $\theta : \mathcal{L}(\mathbb{C}^2) \to \mathcal{L}(\mathbb{C}^2)$ with the initial state $\frac{1}{2} |\Omega\rangle \langle\Omega|$, where $|\Omega\rangle = |00\rangle + |11\rangle$. Consider the action of the group of the local unitaries $\{U_1 \otimes U_2 \mid U_1, U_2 : \mathbb{C}^2 \to \mathbb{C}^2 \text{ are unitaries}\}$ on the set of the Hermitian operators over $\mathbb{C}^4$, which is defined as $X \mapsto (U_1 \otimes U_2)X(U_1 \otimes U_2)^\dagger$. The set of the SDP witnesses obtained from the transpose map with pure states as their initial states is equal to the orbit of $W$ under this group action.*

# Conclusion

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- Recognition of SEP is hard, but we can develop easy necessary conditions.
- Positive but not completely positive maps and entanglement witnesses are two important family of entanglement criteria.
- These two families of criteria are related to each other.
- Semidefinite programming can be used to obtain an optimal family of witnesses.

# Thank you!